

Circular geodesics in the Hartle-Thorne metric

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Abstract. The Hartle-Thorne metric is an exact solution of vacuum Einstein field equations that describes the exterior of any slowly and rigidly rotating, stationary and axially symmetric body. The metric is given with accuracy up to the second order terms in the body's angular momentum, and first order in its quadrupole moment. We give, with the same accuracy, analytic formulae for circular geodesics in the Hartle-Thorne metrics. They describe angular velocity, angular momentum, energy, epicyclic frequencies, shear, vorticity and Fermi-Walker precession. These quantities are relevant to several astrophysical phenomena, in particular to the observed high frequency, kilohertz Quasi Periodic Oscillations (kHz QPOs) in the X-ray luminosity from black hole and neutron star sources. It is believed that kHz QPO data may be used to test the strong field regime of Einstein's general relativity, and the physics of super-dense matter of which neutron stars are made of.

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1. The astrophysical motivation

Thanks to precise timing of binary pulsars' orbital decay [1], the weak field limit of Albert Einstein's general relativity is far better tested than any other physical theory. However, the most interesting and bizarre predictions of Einstein's theory deal not with the weak field, but with the extremely strong field regime and these have never been tested. Thus, the very question [2] "Was Einstein right?" remains unanswered. One may argue that in the foreseeable future there will be no way to test super-strong gravity. Obviously, this cannot be done in laboratories on Earth. Central regions of Galactic sources containing black holes and neutron star do have gravity sufficiently strong for such tests, but they are only several tens of kilometer across and typically observed from kiloparsecs away. Thus, they cannot be spatially resolved with current instruments. However, already existing technology [3] provides very precise resolution in time of the observed variability of X-ray radiation coming from the vicinity of black holes and neutron stars.

Of special importance here are the kilohertz, Quasi Periodic Oscillations (kHz QPOs) of the X-ray flux from galactic low mass X-ray binaries in which the compact object is either a black hole or a neutron star [4]. The QPOs are often observed in twin pairs of kilohertz frequencies. The observed frequency ratios are rational, with the particular ratio 3:2 being most common. In several sources it was possible to detect the orbital binary motion, and deduce the mass of the compact object from Kepler's laws. In a few cases the deduced mass is above the Chandrasekhar limit, suggesting a black hole.

Rational ratios of oscillation frequencies suggest a resonance. There is a strong reason to think of a particular resonance here, the non-linear parametric resonance between the vertical and radial epicyclic oscillations in accretion disks [5]. The world lines of matter in accretion disks differ only slightly from those corresponding to circular geodesics. The 3:2 resonance occurs at a precise location inside the disk, only a few gravitational radii from the source, i.e., in the super-strong gravity regime. This gives a unique opportunity for observational strong-gravity tests. In particular, for three black hole twin kHz QPO sources with known masses, the resonance model has already been used for direct measurements of the black hole spin [6, 7]. That was possible due to the fact that the black hole metric is known analytically (the Kerr solution).

For neutron and strange stars, the exterior metric is not analytically known in general. It is known only in the special but important case of a perfect fluid, stationary and axially symmetric star with mass M_0 , angular momentum J_0 , and quadrupole moment Q_0 , that rotates rigidly and slowly. In this case, the Hartle & Thorne [8] solution gives the metric analytically, with accuracy up to the second order terms in stellar dimensionless angular momentum $j = J/M^2$, and first order in terms of dimensionless quadrupole, $q = -Q/M^3$.

We give here, with the same accuracy, analytic formulae for circular geodesics in the Hartle-Thorne metrics. They describe angular velocity, angular momentum, energy, epicyclic frequencies, shear, vorticity and Fermi-Walker precession. These quantities are of interest in modeling of several astrophysical phenomena, in particular of the kHz QPOs (e.g., [9]). The radius of the innermost stable marginally orbit and related quantities have previously been computed to linear order in j by Klužniak & Wagoner [10], through quadratic and higher orders in j and q by Shibata & Sasaki [11], Sibgatullin & Sunayev [12], and numerically by Berti & Stergioulas [13].

Although the derivation of the following formulae is simple, it is technically troublesome and one must worry about typographical errors. We derived all the formulae in a semi automatic way [14] using Mathematica.

2. The Hartle-Thorne Metric

We use the geometrical units, in which $c = 1 = G$. In these units, mass, length and time are all measured in centimeters. Adding an asterisk (*) as the indicator of the usual physical units, we write for the mass, radius and time in geometrical units, $M = GM^*/c^2$, $r = r^*$, and $t = ct^*$. In these units and with spherical coordinates (t, r, θ, ϕ) , the Hartle-Thorne metric [15, 16, 8] reads,

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2 + g_{\phi t} d\phi dt + g_{t\phi} dt d\phi, \quad (1)$$

where $g_{t\phi} = g_{\phi t}$ and the components of the metric tensor are given below.

$$\begin{aligned} g_{tt} &= + (1 - 2M/r) [1 + j^2 F_1 + q F_2] \\ g_{rr} &= - (1 - 2M/r)^{-1} [1 + j^2 G_1 - q F_2] \\ g_{\theta\theta} &= -r^2 [1 + j^2 H_1 + q H_2] \\ g_{\phi\phi} &= -r^2 \sin^2 \theta [1 + j^2 H_1 + q H_2] \\ g_{t\phi} &= -2(M^2/r) j \sin^2 \theta \end{aligned} \quad (2)$$

where

$$\begin{aligned} F_1 &= [8Mr^4(r - 2M)]^{-1} \cdot \\ &\quad [u^2 (48M^6 - 8M^5r - 24M^4r^2 - 30M^3r^3 - 60M^2r^4 + 135Mr^5 - 45r^6) \\ &\quad + (r - M)(16M^5 + 8M^4r - 10M^2r^3 - 30Mr^4 + 15r^5)] + A_1(r) \end{aligned} \quad (3)$$

$$F_2 = [8Mr(r - 2M)]^{-1} (5(3u^2 - 1)(r - M)(2M^2 + 6Mr - 3r^2)) - A_1(r) \quad (4)$$

$$G_1 = [8Mr^4(r - 2M)]^{-1} ((L - 72M^5r) - 3u^2 (L - 56M^5r)) - A_1(r) \quad (5)$$

$$L = (80M^6 + 8M^4r^2 + 10M^3r^3 + 20M^2r^4 - 45Mr^5 + 15r^6) \quad (6)$$

$$A_1 = \frac{15r(r - 2M)(1 - 3u^2)}{16M^2} \ln \left(\frac{r}{r - 2M} \right) \quad (7)$$

$$H_1 = (8Mr^4)^{-1} (1 - 3u^2)(16M^5 + 8M^4r - 10M^2r^3 + 15Mr^4 + 15r^5) + A_2(r) \quad (8)$$

$$H_2 = (8Mr)^{-1} (5(1 - 3u^2)(2M^2 - 3Mr - 3r^2)) - A_2(r) \quad (9)$$

$$A_2 = \frac{15(r^2 - 2M^2)(3u^2 - 1)}{16M^2} \ln \left(\frac{r}{r - 2M} \right) \quad (10)$$

and where in addition $u := \cos \theta$.

We have checked by directly calculating the Ricci tensor R_{ik} , that the above Hartle-Thorne metric is indeed a solution of the vacuum Einstein's field equations: with the relevant accuracy to quadratic terms in j and linear terms in q , $R_{ik} = 0$.

The Kerr metric in the Boyer-Lindquist coordinates could be obtained from the above Hartle-Thorne metric after putting $a = Mj$, $q = j^2$, and making a coordinate transformation implicitly given by,

$$r_{BL} = r - a^2/(2r^3) ((r + 2M)(r - M) + u^2(r - 2M)(r + 3M)) \quad (11)$$

$$\theta_{BL} = \theta - a^2/(2r^3)(r + 2M) \cos \theta \sin \theta \quad (12)$$

3. The detailed formulae

3.1. The Horizon

The radius of the horizon (if present) is obtained by setting $g_{\phi t}^2 - g_{tt}g_{\phi\phi} = 0$ and solving for r .

$$r_h = 2M \left[1 - \frac{1}{16} j^2 (7 - 15u^2) + \frac{5}{16} q (1 - 3u^2) \right]. \quad (13)$$

3.2. The Ergosphere

Similarly, the radius of ergosphere can be easily obtained by setting $g_{tt} = 0$,

$$r_0 = 2M \left[1 - \frac{1}{16} j^2 (3 - 11u^2) + \frac{5}{16} q (1 - 3u^2) \right] \quad (14)$$

3.3. Dragging of Inertial Frames

The angular velocity of an inertial observer near the rotating star as observed by someone at infinity (i.e. the frame-dragging) is given by:

$$\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2J}{r^3} = \frac{2M^2 j}{r^3}. \quad (15)$$

As such, it is independent of the quadrupole moment and the second order terms as well as the angle of inclination.

3.4. The Orbital Angular Velocity ($\Omega = u^\phi/u^t$)

The angular velocity for corotating/counterrotating circular particle orbits is given by:

$$\Omega = \pm \frac{M^{1/2}}{r^{3/2}} \left[1 \mp j \frac{M^{3/2}}{r^{3/2}} + j^2 F_1(r) + q F_2(r) \right] \quad (16)$$

where

$$F_1(r) = [48M^7 - 80M^6 r + 4M^5 r^2 - 18M^4 r^3 + 40M^3 r^4 + 10M^2 r^5 \quad (17)$$

$$+ 15M r^6 - 15r^7] (16M^2 (r - 2M) r^4)^{-1} + A(r) \quad (18)$$

$$F_2(r) = \frac{5(6M^4 - 8M^3 r - 2M^2 r^2 - 3Mr^3 + 3r^4)}{16M^2 (r - 2M) r} - A(r) \quad (19)$$

$$A(r) = \frac{15(r^3 - 2M^3)}{32M^3} \ln \left(\frac{r}{r - 2M} \right) \quad (20)$$

3.5. The Specific Angular Momentum ($\ell = -u_\phi/u_t$)

$$\ell = \pm \ell_0 \left[1 \mp j \frac{M^{3/2}(3r - 4M)}{r^{3/2}(r - 2M)} + j^2 F_1(r) - q F_2(r) \right] \quad (21)$$

where

$$\ell_0 := \frac{M^{1/2}r^{3/2}}{r - 2M} \quad (22)$$

$$\begin{aligned} F_1(r) = & \left[16M^2 r^4 (r - 2M)^2 \right]^{-1} (96M^8 - 112M^7r - 8M^6r^2 - 48M^5r^3 \\ & + 42M^4r^4 + 220M^3r^5 - 260M^2r^6 + 105Mr^7 - 15r^8) + A(r) \end{aligned} \quad (23)$$

$$F_2(r) = [16M^2 r (r - 2M)]^{-1} 5 (6M^4 - 22M^2r^2 + 15Mr^3 - 3r^4) + A(r) \quad (24)$$

$$A(r) = \frac{15}{32M^3} (2M^3 + 4M^2r - 4Mr^2 + r^3) \ln \left(\frac{r}{r - 2M} \right) \quad (25)$$

3.6. The Specific Energy ($\varepsilon = u_t$)

$$\varepsilon = E_0 [1 \mp j F_1(r) + j^2 F_2(r) + q F_3(r)] \quad (26)$$

where

$$E_0 := \frac{r - 2M}{r^{1/2}(r - 3M)^{1/2}} \quad (27)$$

$$F_1(r) = \frac{M^{5/2}}{r^{1/2}(r - 2M)(r - 3M)} \quad (28)$$

$$\begin{aligned} F_2(r) = & \left[16M r^4 (r - 2M) (r - 3M)^2 \right]^{-1} (144M^8 - 144M^7r - 28M^6r^2 \\ & - 58M^5r^3 - 176M^4r^4 + 685M^3r^5 - 610M^2r^6 + 225Mr^7 - 30r^8) + B(r) \end{aligned} \quad (29)$$

$$F_3(r) = \frac{5(r - M)(6M^3 - 20M^2r - 21Mr^2 + 6r^3)}{16Mr(r - 2M)(r - 3M)} - B(r) \quad (30)$$

$$B(r) = \frac{15r(8M^2 - 7Mr + 2r^2)}{32M^2(r - 3M)} \ln \left(\frac{r}{r - 2M} \right) \quad (31)$$

3.7. Radius of marginally stable, marginally bound and photon orbit.

The condition $\varepsilon = u_t = 1$ gives the radius of the marginally bound orbit, r_{mb} and the condition $u^t = 0$ gives the photon orbit, r_{ph} . In addition by setting $d\ell/dr = 0$ we can solve for the radius of the marginally stable orbit, r_{ms} .

$$r_{mb} = 4M [1 \mp \frac{1}{2}j - j^2 (8047/256 - 45 \ln 2) + q (1005/32 - 45 \ln 2)] \quad (32)$$

$$r_{ph} = 3M \left[1 \mp j \frac{2\sqrt{3}}{9} - j^2 \left(\frac{7036 - 6075 \ln 3}{1296} \right) + q \left(\frac{7020 - 6075 \ln 3}{1296} \right) \right] \quad (33)$$

$$r_{ms} = 6M \left[1 \mp j \frac{2}{3}\sqrt{\frac{2}{3}} + j^2 \left(\frac{251647}{2592} - 240 \ln \frac{3}{2} \right) + q \left(-\frac{9325}{96} + 240 \ln \frac{3}{2} \right) \right] \quad (34)$$

3.8. The Epicyclic Frequencies

We define an effective potential as: $U(r, \theta, \ell) := g^{tt} - 2\ell g^{t\phi} + \ell^2 g^{\phi\phi}$, which can be used to find the general formula for the epicyclic frequencies on the equatorial plane.

$$\tilde{\kappa}_x^2 = \frac{(g_{tt} + \Omega_\pm g_{t\phi})^2}{2g_{xx}} \left(\frac{\partial^2 U}{\partial x^2} \right)_\ell, \quad x \in (r, \theta) \quad (35)$$

Then we have:

$$\tilde{\kappa}_r^2 = M(r - 6M)r^{-4} [1 \pm j F_1(r) - j^2 F_2(r) - q F_3(r)] \quad (36)$$

$$\tilde{\kappa}_\theta^2 = Mr^{-3} [1 \mp j G_1(r) + j^2 G_2(r) + q G_3(r)] \quad (37)$$

where

$$F_1(r) = \frac{6M^{3/2}(r + 2M)}{r^{3/2}(r - 6M)} \quad (38)$$

$$F_2(r) = [8M^2 r^4 (r - 2M) (r - 6M)]^{-1} [384M^8 - 720M^7r - 112M^6r^2 - 76M^5r^3 - 138M^4r^4 - 130M^3r^5 + 635M^2r^6 - 375Mr^7 + 60r^8] + A(r) \quad (39)$$

$$F_3(r) = \frac{5(48M^5 + 30M^4r + 26M^3r^2 - 127M^2r^3 + 75Mr^4 - 12r^5)}{8M^2 r (r - 2M) (r - 6M)} - A(r) \quad (40)$$

$$A(r) = \frac{15r(r - 2M)(2M^2 + 13Mr - 4r^2)}{16M^3(r - 6M)} \ln \left(\frac{r}{r - 2M} \right) \quad (41)$$

$$G_1(r) = \frac{6M^{3/2}}{r^{3/2}} \quad (42)$$

$$G_2(r) = [8M^2 r^4 (r - 2M)]^{-1} [48M^7 - 224M^6r + 28M^5r^2 + 6M^4r^3 - 170M^3r^4 + 295M^2r^5 - 165Mr^6 + 30r^7] - B(r) \quad (43)$$

$$G_3(r) = \frac{5(6M^4 + 34M^3r - 59M^2r^2 + 33Mr^3 - 6r^4)}{8M^2 r (r - 2M)} + B(r) \quad (44)$$

$$B(r) = \frac{15(2r - M)(r - 2M)^2}{16M^3} \ln \left(\frac{r}{r - 2M} \right) \quad (45)$$

3.9. Shear and Vorticity

The general formulae for shear and vorticity are [17]:

$$\sigma^2 = -\frac{1}{4}(1 - \Omega\ell)^{-2}\mathcal{R}^2(\nabla_a\Omega)(\nabla^a\Omega) \quad (46)$$

$$\omega^2 = -\frac{1}{4}(1 - \Omega\ell)^{-2}\mathcal{R}^{-2}(\nabla_a\ell)(\nabla^a\ell) \quad (47)$$

$$\mathcal{R}^2 := \left(\frac{U}{E} \right)^2 (g_{t\phi}^2 - g_{tt}g_{\phi\phi}) = \frac{(\ell g_{t\phi} + g_{\phi\phi})^2}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}} \quad (48)$$

The results for the HT metric are:

$$\sigma^2 = S_0 [1 \pm j F_1(r) + j^2 F_2(r) + q F_3(r)] \quad (49)$$

$$\omega^2 = V_0 [1 \mp j G_1(r) + j^2 G_2(r) + q G_3(r)] \quad (50)$$

where

$$S_0 := \frac{9M}{16r^3} \frac{(r - 2M)^2}{(r - 3M)^2} \quad (51)$$

$$V_0 := \frac{M}{16r^3} \frac{(r - 6M)^2}{(r - 3M)^2} \quad (52)$$

and

$$F_1(r) = \frac{4M^{3/2}}{r^{1/2}(r - 3M)} \quad (53)$$

$$\begin{aligned} F_2(r) = & [8M^2r^4(r-2M)^2(r-3M)^2]^{-1} \cdot \\ & - (864M^{10} - 1776M^9r + 1048M^8r^2 - 592M^7r^3 - 10M^6r^4 + 542M^5r^5 \\ & - 693M^4r^6 + 820M^3r^7 - 530M^2r^8 + 150Mr^9 - 15r^{10}) - A(r) \end{aligned} \quad (54)$$

$$F_3(r) = \frac{5(36M^6 - 2M^5r - 28M^4r^2 + 35M^3r^3 - 43M^2r^4 + 21Mr^5 - 3r^6)}{8M^2r(r-2M)^2(r-3M)} + A(r) \quad (55)$$

$$A(r) = \frac{15r(r-4M)(M^2+r^2)}{16M^3(r-3M)} \ln \left(\frac{r}{r-2M} \right) \quad (56)$$

$$G_1(r) = \frac{12M^{3/2}(r-2M)}{r^{1/2}(r-3M)(r-6M)} \quad (57)$$

$$\begin{aligned} G_2(r) = & - (7776M^{10} - 12528M^9r + 3672M^8r^2 - 4080M^7r^3 + 150M^6r^4 \\ & - 12594M^5r^5 + 30891M^4r^6 - 25620M^3r^7 + 9590M^2r^8 - 1650Mr^9 \\ & + 105r^{10}) \cdot [8M^2r^4(r-3M)^2(r-6M)^2]^{-1} + B(r) \end{aligned} \quad (58)$$

$$G_3(r) = \frac{-5(54M^5 + 30M^4r + 147M^3r^2 - 271M^2r^3 + 141Mr^4 - 21r^5)}{8M^2r(r-3M)(r-6M)} - B(r) \quad (59)$$

$$B(r) = \frac{15r(24M^4 - 126M^3r + 135M^2r^2 - 54Mr^3 + 7r^4)}{16M^3(r-3M)(r-6M)} \ln \left(\frac{r}{r-2M} \right) \quad (60)$$

3.10. The Fermi-Walker Precession

The precession for a gyroscope on a general circular orbit in a general stationary and axially symmetric spacetime was calculated in [18]. In the equatorial plane for the Hartle-Thorne metric we use formula (4.8) of [18] to obtain:

$$\Pi = \Pi_0 [-1 \mp j F_1(r) + j^2 F_2(r) + q F_3(r)] \quad (61)$$

$$\Pi_0 := \frac{M^{1/2}r^2}{(r-3M)^{1/2}(r-2M)^3} \quad (62)$$

$$F_1(r) = \frac{3M^{5/2}(r-4M)}{2r^{3/2}(r-2M)(r-3M)} \quad (63)$$

$$\begin{aligned} F_2(r) = & (576M^{10} - 1704M^9r + 2904M^8r^2 - 1280M^7r^3 + 1604M^6r^4 - 8060M^5r^5 \\ & + 10688M^4r^6 - 6290M^3r^7 + 1865M^2r^8 - 270Mr^9 + 15r^{10}) \cdot \\ & [16M^2r^4(r-2M)^2(r-3M)^2]^{-1} - A(r) \end{aligned} \quad (64)$$

$$F_3(r) = \frac{5(12M^4 + 36M^3r - 94M^2r^2 + 33Mr^3 - 3r^4)}{16M^2r(r-3M)} + A(r) \quad (65)$$

$$A(r) = \frac{15r(r-4M)(10M^2 - 8Mr + r^2)}{32M^3(r-3M)} \ln \left(\frac{r}{r-2M} \right) \quad (66)$$

3.11. Resonance

Assuming an $m : n$ resonance between epicyclic frequencies, we calculate the radius for which the resonance occurs given the condition that $m\tilde{\kappa}_r = n\tilde{\kappa}_\theta$ and denoting

$$z := m^2/n^2,$$

$$r_{mn} = r_0 [1 - j F_1(m, n) + j^2 F_2(m, n) + q F_3(m, n)] \quad (67)$$

where

$$r_0 = 6 M \left(1 - \frac{n^2}{m^2}\right)^{-1} \quad (68)$$

and

$$F_1 = \frac{1}{m^3} \left(\frac{2}{3^3}\right)^{1/2} (m^2 - n^2)^{1/2} (2m^2 + n^2) \quad (69)$$

$$\begin{aligned} F_2 &= (1 - 50z - 11z^2 + 385z^3 + 10612z^4 + 123286z^5 + 496927z^6 \\ &\quad + 691843z^7 + 251647z^8) \cdot \left[1296z^4(z-1)^3(1+2z)\right]^{-1} - A(z) \end{aligned} \quad (70)$$

$$F_3 = \frac{-5(1+5z)(1+74z+546z^2+950z^3+373z^4)}{48(z-1)^3z(1+2z)} + A(z) \quad (71)$$

$$A(z) = \frac{15(1+2z)(1+12z+63z^2+32z^3)}{4(z-1)^4} \ln\left(\frac{3z}{1+2z}\right) \quad (72)$$

4. Software and hardware

These results were all derived and checked using Mathematica 4.0 [19] running on a PIII 800 MHz/256MB under Linux Red Hat 7.2. In addition the packages TTC [20] and GRTensorM [21] were found useful. The Mathematica notebooks containing the calculations can be found on:

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